Journal of Statistical Physics, Vol. 11, No. 2, 1974

Approach to Two Dimensions in the n-Vector Model

Daniel J. Amit[1] and Marco Zannetti[2]

Received April 8, 1974

The specific heat of the n-vector model to lowest order in $1/n$ is analyzed when the dimensionality varies continuously between $d = 3$ and $d = 2$. The changeover from three-dimensional to two-dimensional behavior is related to the location of the bare transition temperature with respect to the critical region.

KEY WORDS: Phase transitions; specific heat; two-dimensional systems; broken continuous symmetry.

1. INTRODUCTION

In two-dimensional systems with a vector order parameter there is no phase transition to a state with long-range order.[1]

On the other hand, model calculations[2,3] (self-consistent Hartree approximation) for superconducting films exhibit an anomaly in the specific heat at a finite temperature, which seems to agree with experiment.[4] The relevance of this kind of approximation to the description of superconducting films has been put in doubt by the more accurate results of Scalapino *et al.*[5] However, in their calculation the specific heat anomaly is even more pronounced then in the Hartree approximation.

Here we consider the n-vector model in the limit $n \to \infty$, which is equivalent to the Hartree approximation,[5] leaving aside the question of a realistic description of superconducting films. Our concern is the interpretation of the specific heat anomaly in a model which does not undergo a phase transition.

In order to explain the origin of the specific heat anomaly we regard the two-dimensional case as the limiting behavior obtained when the dimensionality is allowed to vary continuously from $d = 3$ to $d = 2$. This device

[1] Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem, Israel.
[2] Istituto di Fisica, Universita di Salerno, Salerno, Italy.

allows one to separate clearly the effect of dimensionality from the fluctuation effects due to the interaction.

We recall that when order parameter fluctuations are neglected the theory reduces to mean-field theory. The results of mean-field theory are independent of dimensionality. In particular, the transition occurs at a value of the temperature which is the same whatever the dimensionality of the system.

If one takes the fluctuations into account, the transition temperature is shifted to a lower value. Furthermore, there is an interval around the transition temperature (critical region) where the behavior of the system deviates from the predictions of mean-field theory.

In this paper we shall pay particular attention to the relationship between the transition temperature shift and the size of the critical region. In three dimensions the former is always smaller than the latter.[6] Consequently, mean-field behavior is obliterated by the fluctuations. However, when the dimensionality is lowered the opposite situation may occur, and indeed it always occurs in two dimensions. In this case the transition temperature shift exceeds the size of the critical region and mean-field behavior competes with critical behavior. We ascribe to this fact the qualitative features of two-dimensional specific heat.

2. THE MODEL

We consider a system described by an n-component Ginzburg–Landau (GL) field $\boldsymbol{\varphi}(\mathbf{x}) = [\varphi_1(\mathbf{x}),\ldots, \varphi_n(\mathbf{x})]$. Namely, the partition function is written as a functional integral over all order parameter configurations

$$Z = \int \mathscr{D}[\varphi] e^{-\beta \mathscr{H}[\varphi]} \tag{1}$$

where $\beta = (K_{\mathrm{B}}T)^{-1}$. Here $\mathscr{H}[\varphi]$ is the GL functional generalized to the case of an n-component field[5]

$$\mathscr{H}[\varphi] = \tfrac{1}{2} \int d^d x [(\nabla\varphi)^2 + r_0\varphi^2 + (u/2n)(\varphi^2)^2] \tag{2}$$

Here

$$(\nabla\varphi)^2 = \sum_{i=1}^{n} (\nabla\varphi_i)^2 \tag{3}$$

$$\varphi^2 = \sum_{i=1}^{n} (\varphi_i)^2 \tag{4}$$

and d is the dimensionality of the system. In the GL theory r_0 is a regular function of the temperature

$$r_0 = a(T - T_0) \tag{5}$$

where a is a positive constant and T_0 is the GL transition temperature.

Usually the inverse temperature β is absorbed in the energy functional by rescaling the fields[6]

$$\beta\varphi^2 = \phi^2 \tag{6}$$

This implies

$$r_0 \rightarrow r_0, \qquad u \rightarrow \beta^{-1}u \tag{7}$$

The temperature dependence that one introduces in the coupling constant is considered irrelevant to the critical behavior. This is correct as long as the transition temperature is not close to $T = 0$. However, when $d \rightarrow 2$ the transition temperature does tend to zero. Consequently, in order to study the limit $d \rightarrow 2$, we keep explicitly all temperature dependences.

The central quantity in the following analysis is the order parameter correlation function

$$G(\mathbf{x}) = \langle \varphi_i(\mathbf{x})\varphi_i(0)\rangle = Z^{-1}\int \mathscr{D}[\varphi]\varphi_i(\mathbf{x})\varphi_i(0)e^{-\beta\mathscr{H}[\phi]} \tag{8}$$

which satisfies Dyson's equation.[7] Switching to Fourier space, this takes the form

$$G^{-1}(\mathbf{k}) = G_0^{-1}(\mathbf{k}) - \Sigma(\mathbf{k}) \tag{9}$$

with

$$G_0^{-1}(\mathbf{k}) = \beta(k^2 + r_0) \tag{10}$$

Once the order parameter correlation function is known, the thermodynamic properties of the system can be derived from the free energy density written as a functional of $G(\mathbf{k})$,[7]

$$W[G] = \beta^{-1}\frac{n}{2}\int \frac{d^dk}{(2\pi)^d}\, G_0^{-1}(\mathbf{k})G(\mathbf{k}) - \frac{1}{2}\beta^{-1}n\int \frac{d^dk}{(2\pi)^d}\,\log G(\mathbf{k}) - \frac{\beta^{-1}}{2}\,\Phi[G] \tag{11}$$

The functional $\Phi[G]$, which is related to the self-energy by

$$\delta\Phi/\delta G(\mathbf{k}) = \Sigma(\mathbf{k}) \tag{12}$$

is given by the sum of all connected diagrams with no external legs.

The leading term in the $1/n$ expansion of the self-energy, namely the limit of Σ when $n \rightarrow \infty$, is given by the sum of the tree diagrams.[8] To this order, the approximation is self-consistent,[9] that is

$$\Sigma = -u\beta N + O(1/n) \tag{13}$$

with

$$N = \int \frac{d^dk}{(2\pi)^d}\, G(\mathbf{k}) \tag{14}$$

and $G(\mathbf{k})$ must be solved for from Dyson's equation. In this limit the theory gives the same critical behavior as the spherical model.[10]

3. TRANSITION TEMPERATURE AND CRITICAL REGION

Since to lowest order the correction to the free propagator in Eq. (9) is independent of k, $G(k)$ will be of the form

$$G^{-1}(k) = \beta(k^2 + \kappa^2) \tag{15}$$

where κ, the inverse correlation length, is given by

$$\kappa^2 = r_0 + uN \tag{16}$$

Equation (16) is an implicit equation for κ, where

$$N = \beta^{-1} S_d \int_0^\Lambda dk \frac{k^{d-1}}{k^2 + \kappa} = \beta^{-1} S_d \left[\frac{\Lambda^\sigma}{\sigma} - F(\sigma)\kappa^\sigma \right] \tag{17}$$

with

$$\sigma = d - 2, \qquad S_d = \frac{2}{2^d \pi^{d/2} \Gamma(d/2)}, \qquad F(\sigma) = \frac{\pi}{2 \sin(\frac{1}{2}\pi\sigma)} \tag{18}$$

and Λ is a momentum cutoff introduced to keep the integral finite. (See, e.g., Ref. 6.) At T_c (transition temperature) we expect κ to vanish. Hence Eq. (16) gives

$$r_{0c} + wT_c(\Lambda^\sigma/\sigma) = 0 \tag{19}$$

where $r_{0c} = a(T_c - T_0)$ and $w = uK_B S_d$. Solving for T_c, we get

$$T_c = aT_0/[a + w(\Lambda^\sigma/\sigma)] \tag{20}$$

From this one sees that $T_c < T_0$, namely the transition temperature is depressed, and that

$$\lim_{\sigma \to 0} T_c = 0 \tag{21}$$

From dimensional considerations on Eq. (2) one can immediately derive that the coupling constant u has dimensions energy $\times$ (length)$^{d-4}$.[6] Thus one can construct Ferrell's length[11]

$$R = (wT_0)^{1/(d-4)} \tag{22}$$

and Ginzburg's criterion[6,12] can be restated in the following form: The effect of fluctuations can be neglected as long as the correlation length is smaller than R.

Since in the GL theory the correlation length is given by $r_0^{-1/2}$, the order of magnitude of the size of the critical region will be determined by the condition $r_0^{-1/2} = R$, which we write in the form

$$\xi_0^{-2} \epsilon_0 = R^{-2} \tag{23}$$

where $\xi_0 = (aT_0)^{-1/2}$ is the GL correlation length at $T = 0$ and ϵ_0 is the size of the critical region in units of T_0.

Returning to Eq. (20), the shift of the transition temperature in units of T_0 is given by

$$\Delta\epsilon = \frac{T_0 - T_c}{T_0} = \frac{\epsilon_0(R\Lambda)^\sigma}{\sigma + \epsilon_0(R\Lambda)^\sigma} \tag{24}$$

Away from criticality, the only fundamental length in the problem is Λ^{-1}, hence R will be of the order of Λ^{-1}. From Eq. (24) then we get

$$\Delta\epsilon/\epsilon_0 \simeq 1/(\sigma + \epsilon_0) \tag{25}$$

In the three-dimensional case ($\sigma = 1$) we have, therefore, that $\Delta\epsilon$ is always smaller than ϵ_0, no matter how we choose the coupling constant.

On the other hand, since in the chosen units $\epsilon_0 < 1$, when we reach the two-dimensional limit ($\sigma = 0$), $\Delta\epsilon$ is always larger than ϵ_0. In the intermediate cases both possibilities can occur, namely

$$\Delta\epsilon/\epsilon_0 \gtrless 1 \qquad \text{for} \quad \sigma \gtrless 1 - \epsilon_0 \tag{26}$$

$\Delta\epsilon < \epsilon_0$ means that the GL transition temperature T_0 is inside the critical region, while $\Delta\epsilon > \epsilon_0$ means that T_0 is outside. We expect the behavior of the system to be substantially different in the two cases. In fact, while in the first case the fluctuations cancel any trace of mean-field behavior, in the second case mean-field behavior competes with critical behavior. $\sigma = 1$ and $\sigma = 0$ are the limiting cases in which the above two types of behavior are dominant, respectively. In the intermediate situations we say that the system exhibits three-dimensional or two-dimensional behavior according to $\sigma > 1 - \epsilon_0$ or $\sigma < 1 - \epsilon_0$.

It should be remarked at this point that the above considerations seem to be in contrast to the idea of universality, which wants critical behavior to depend only on the space dimensionality and on the vector dimensionality of the order parameter, while we say that at a fixed σ the behavior of the system can depend on ϵ_0, namely on the coupling constant. However, our considerations refer to the specific heat, which in this model remains finite. Namely the energy density does not fluctuate strongly. Furthermore, as will become clear from the following, the anomalies observed when mean-field theory competes with critical behavior occur away from the transition temperature and are not associated with divergences.

4. CORRELATION LENGTH AND SPECIFIC HEAT

To leading order the functional Φ is given by

$$\Phi = -\tfrac{1}{2}nu\beta N^2 + O(1) \tag{27}$$

Inserting this in Eq. (11), we get the free energy density

$$W = \frac{n}{2}\int \frac{d^dk}{(2\pi)^d}(\beta^{-1}) - \frac{unN^2}{4} - \frac{n}{2}\beta^{-1}\frac{d^dk}{(2\pi)^d}\log G(\mathbf{k}) \tag{28}$$

Next, we differentiate twice with respect to the temperature and we obtain the specific heat per degree of freedom, which we write in dimensionless form as

$$C = \Delta C_0\left[\frac{T}{T_0} - \frac{T_0}{T_c}(\xi_0{}^2\kappa^2 + 1)\xi_0{}^2\frac{d\kappa^2}{d\tau}\right] \tag{29}$$

where

$$\Delta C_0 = a^2T_0/2w\Lambda^d \tag{30}$$

is the GL specific heat jump and

$$\tau = (T - T_c)/T_c \tag{31}$$

In order to compute the specific heat one must first solve Eq. (16) for κ^2. Using Eq. (20), this can be written in the form

$$\kappa^2 + TwF(\sigma)\kappa^\sigma = \xi_0^{-2}\tau \tag{32}$$

From the above equation, in the limit of small τ, we get $\kappa^\sigma\alpha\tau$, that is, we recover the known results for the critical indices of the correlation length and susceptibility[3]

$$\nu = 1/(d-2), \qquad \gamma = 2/(d-2) \tag{33}$$

When the limit $\sigma \to 0$ is taken both the r.h.s. and the l.h.s. of Eq. (32) diverge; however, the divergences cancel out. It is convenient to introduce the temperature

$$T^* = T_0/[1 + (w/a)\log\Lambda] \tag{34}$$

The limit $\sigma \to 0$ of Eq. (32) can then be written in the form

$$\kappa^2 + Tw\log\kappa = \xi_0^{-2}(T - T^*)/T^* \tag{35}$$

This does not mean that in two dimensions the transition temperature is T^*. In fact, at T^*, κ is finite. However, for T small Eq. (35) is approximated by

$$\log\kappa = -aT_0/wT \tag{36}$$

which indicates that κ vanishes at $T = 0$, faster than any power of T, in agreement with Eq. (21). Namely,

$$\kappa = \exp(-aT_0/wT) \tag{37}$$

The equation for the specific heat remains unaltered except for the substitution of T^* for T_c.

[3] In this model the susceptibility is proportional to κ^{-2}.

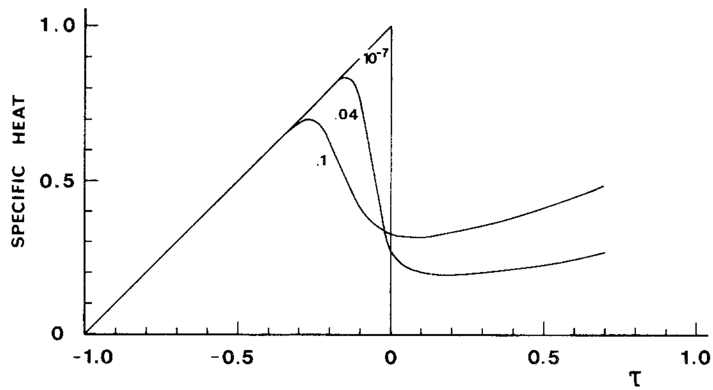

Fig. 1. Normalized specific heat $C/[(\Delta C_0)^{T^*}_{T_0}]$ versus reduced temperature $\tau = (T - T^*)/T^*$ in the two dimensions. The curves are labeled by the size of the critical region.

The results of the numerical calculation are displayed in Figs. 1 and 2. Figure 1 is a plot of the normalized specific heat in the two-dimensional case for different values of ϵ_0. When ϵ_0 is very small the specific heat behaves as in the GL theory. If ϵ_0 grows, the curve of the specific heat deviates from the GL curve in the neighborhood of T_0. However, the qualitative behavior remains unchanged. The shift of the peak corresponds to a shift in the same direction of the bare transition temperature with respect to T^*. In fact, in the units we have chosen $(T_0 - T^*)/T^* = \frac{1}{2}\epsilon_0 \log \epsilon_0$, with $\epsilon_0 < 1$.

Next, we let the dimensionality vary. The value of ϵ_0 is kept fixed in order to see the change in behavior when σ crosses the value $1 - \epsilon_0$. This

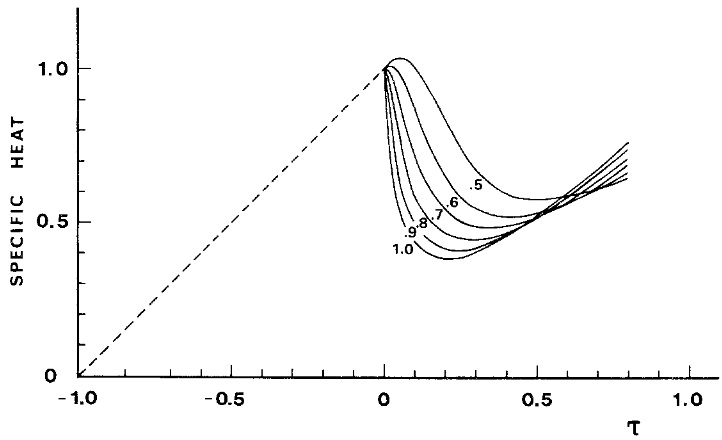

Fig. 2. Normalized specific heat $C/[(\Delta C_0)T_c/T_0]$ versus reduced temperature $T = (T - T_c)/T_c$ at fixed value of the size of the critical region $\epsilon_0 = 0.3$. The curves are labeled by $\sigma = d - 2$. The dashed line is the specific heat in the ordered phase.

indeed occurs in Fig. 2. For $\sigma > 1 - \epsilon_0$ there is a cusp at T_c as in the three-dimensional case, while for $\sigma < 1 - \epsilon_0$ there is a peak at a value of the temperature intermediate between T_c and T_0 of the type observed in two dimensions. To complete the picture, we may solve for the specific heat below T_c. In fact, to lowest order in $1/n$ this model can be solved both in the ordered and disordered phases.[13] This gives the dashed line in Fig. 2 (see appendix). When we look over the whole temperature range, the specific heat for $\sigma < 1 - \epsilon_0$ behaves exactly as in two dimensions.

In conclusion we ascribe the anomaly in the specific heat observed at $d = 2$ to a competition between critical behavior and mean field behavior. This occurs because in two dimensions the bare transition temperature always lies outside the critical region. This interpretation is supported by the behavior of the specific heat at higher dimensionality, where a changeover from three-dimensional to two-dimensional behavior can be produced by adjusting the parameters in such a way that T_0 is either inside or outside the critical region.

APPENDIX

When there is symmetry breaking we have a nonvanishing value for the average order parameter $\langle\varphi\rangle$, a propagator for the longitudinal modes G_L, and a propagator for the transverse modes G_T.

These quantities satisfy the following set of equations[(13)]:

$$G_0^{-1}(k = 0) = \mu + K \tag{A.1}$$

$$G_L^{-1}(k) = G_0^{-1}(k) - \Sigma_L(k) \tag{A.2}$$

$$G_T^{-1}(k) = G_0^{-1}(k) - \Sigma_T(k) \tag{A.3}$$

where K, Σ_L, Σ_T are functionals of $\langle\varphi\rangle$, G_L, and G_T. Here μ is an external field and $G_0^{-1}(k)$ is given by Eq. (10).

To lowest order in $1/n$, one can construct an approximation for K, Σ_L, and Σ_T which leads to a self-consistent soluble problem, which at the same time is the continuation to the ordered phase of the model considered in this paper.[(13)]

In the absence of an external field Eqs. (A.1)–(A.3) reduce to

$$\langle\varphi\rangle^2 = -(n/u)(r_0 + uN) \tag{A.4}$$

$$G_T^{-1}(k) = G_L^{-1}(k) = \beta k^2 \tag{A.5}$$

with

$$N = \beta^{-1}\Lambda^\sigma/\sigma \tag{A.6}$$

The corresponding free energy density is

$$W = \frac{r_0}{2}\langle\varphi\rangle^2 + \frac{u}{4n}\langle\varphi\rangle^4 - \frac{nuN^2}{4} - \frac{n\beta^{-1}}{2}\int\frac{d^dk}{(2\pi)^d}\log G_T(k) \tag{A.7}$$

After double differentiation with respect to the temperature, the following expression for the specific heat is obtained:

$$C = (\Delta C_0)T/T_0 \tag{A.8}$$

Thus, for $T < T_c$ the specific heat is the same as in the GL theory.

ACKNOWLEDGMENTS

Part of this work was done when both authors were at the Service de Physique Theorique, C.E.N.—Saclay. They are both very grateful for the hospitality shown to them there. One of us (M.Z.) wishes to thank Dr. W. Theumann for useful discussions.

REFERENCES

1. N. D. Mermin and H. Wagner, *Phys. Rev. Letters* **22**:1133 (1966).
2. L. Gunther and L. W. Gruenberg, *Solid State Commun.* **10**:567 (1972).
3. S. Grossmann, P. H. Richter, and C. Wissel, *Solid State Commun.* **11**:433 (1972).
4. G. D. Zally and J. M. Mochel, *Phys. Rev. Letters* **27**:1710 (1971); *Phys. Rev. B* **6** (1972).
5. D. J. Scalapino, R. A. Ferrell, and A. J. Bray, *Phys. Rev. Letters* **31**:292 (1973).
6. D. J. Amit, "The Ginzburg Criterion Rationalized," *J. Phys. C.: Solid State* **7** (1974).
7. D. J. Amit and M. Zannetti, *J. Stat. Phys.* **7**:31 (1973).
8. S. K. Ma, La Jolla Preprint (1973).
9. D. J. Amit and M. Zannetti, *J. Stat. Phys.* **9**:1 (1973).
10. E. Brezin and D. J. Wallace, *Phys. Rev. B* **7** (1973).
11. R. A. Ferrell, in *Contemporary Physics*, Vol. I, IAEA, Vienna (1969).
12. V. L. Ginzburg, *Fiz. Tverd. Tela* **2**:2031 (1960) [English Transl.: *Soviet Phys.—Solid State* **2**:1824 (1960)].
13. M. Zannetti, Saclay Preprint D.Ph-T/73/53.